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Initial-Boundary Value Problem with Dirichlet and Wentzell Conditions for a Mildly Quasilinear Biwave Equation

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Abstract

For a nonstrictly hyperbolic mildly quasilinear biwave equation in the first quadrant, an initial-boundary value problem with the Cauchy conditions specified on the spatial half-line and the Dirichlet and Wentzell conditions applied on the time half-line was examined. The solution was constructed in an implicit analytical form as a solution of some integro-differential equations. The solvability of these equations was investigated using the parameter continuation method. For the problem under study, the uniqueness of the solution was proved, and the conditions under which its classical solution exists were established. In the case when the data were not smooth enough, a mild solution was constructed.

Keywords: method of characteristics, mildly quasilinear biwave equation, nonlinear equation, nonstrictly hyperbolic equation, initial-boundary value problem, matching conditions, classical solution, parameter continuation method, mild solution

Introduction

The classical linear biwave equation

$$(\partial_t^2 - a^2\Delta)(\partial_t^2 - b^2\Delta)u(t,\mathbf{x}) = f(t,\mathbf{x}) \tag{1}$$

applies to mathematical models related to the mathematical theory of elasticity. For example, the Cauchy–Kovalevski–Somigliana solution of the elastodynamic wave equation can be obtained by solving the biwave equation [1]. The Cauchy problem for Eq. (1) was examined in [1, 2] for the cases $a \neq b$ and $a = b$, respectively.

The following equation is one of the simplest one-dimensional linear generalizations of Eq. (1)

$$(\partial_t^2 - a^2\partial_x^2)(\partial_t^2 - b^2\partial_x)u(t,x) + m^2\partial_t^2 u(t,x) = f(t,x), \tag{2}$$

on which the Timoshenko–Ehrenfest beam theory relies [3]. When the axial effect is considered, the equation becomes [4]

$$(\partial_t^2 - a^2\partial_x^2)(\partial_t^2 - b^2\partial_x)u(t,x) + m^2\partial_t^2 u(t,x) + N\partial_x^2 u(t,x) = f(t,x), \tag{3}$$

taking the place of (2).

A large class of boundary value problems was investigated in [5–9] for the linear generalization of Eq. (1) expressed as

$$(\partial_t^2 - a^2\Delta)(\partial_t^2 - b^2\Delta)u(t,\mathbf{x}) + \sum_{|\boldsymbol{\alpha}|\leqslant 3} a^{(\boldsymbol{\alpha})}(\mathbf{x})\mathbf{D}^{\boldsymbol{\alpha}}u(t,\mathbf{x}) = f(t,\mathbf{x}).$$

In [10], it was proposed to describe various physical processes by nonlinear equations of the form

$$(\partial_t^2 - \Delta)^l u(t,\mathbf{x}) = F(u(t,\mathbf{x}), \Delta u(t,\mathbf{x})). \tag{4}$$

For $l = 1$ and $F(u,w) = F(u)$, Eq. (4) reduces to the standard nonlinear wave equation $(\partial_t^2 - \Delta)u(t,x) = F(u(t,x))$, which describes a scalar, spinless, and uncharged particle in the quantum field theory [11]. The symmetry properties of Eq. (4) with $l = 2$ and $F(u,w) = F(u)$ were studied in [11]. The solvability of boundary value problems for Eq. (4) was analyzed using the Leray–Schauder fixed point theorem in [12–15].

All of Eqs. (1)–(4), where $l = 2$, in the one-dimensional case, represent a special instance of the following equation:

$$(\partial_t^2 - a^2\partial_x^2)(\partial_t^2 - b^2\partial_x)u(t,x) = f(t,x,u(t,x),\partial_t u(t,x),\partial_x u(t,x),\\ \partial_t^2 u(t,x),\partial_t\partial_x u(t,x),\partial_x^2 u(t,x)), \tag{5}$$

which is classified as (strictly) hyperbolic if $a \neq b$ and as nonstrictly hyperbolic if $a = b$.

This article focuses on the nonstrictly hyperbolic case of Eq. (5). Section 1 contains a statement of the initial-boundary value problem. In Section 2, this problem is reduced to the solution of integro-differential equations, and their solvability, uniqueness, and well-posedness are established. In Section 3, the existence and uniqueness theorem for the initial-boundary value problem is formulated. Section 4 outlines a mild solution and proves its existence and uniqueness. The last section summarizes the findings of the study.

1. Statement of the Problem

In the domain $Q = (0,\infty) \times (0,\infty)$ of two independent variables $(t,x) \in \overline{Q} \in \mathbb{R}^2$, the following one-dimensional nonlinear equation is considered:

$$(\partial_t^2 - a^2\partial_x^2)^2 u(t,x) = \mathcal{F}[u](t,x) := f(t,x,u(t,x),\partial_t u(t,x),\partial_x u(t,x),\\ \partial_t^2 u(t,x),\partial_t\partial_x u(t,x),\partial_x^2 u(t,x)), \tag{6}$$

where $a > 0$ for definiteness, and f is a function defined on the set $[0,\infty)\times[0,\infty)\times\mathbb{R}^6$. Equation (6) is equipped with the initial conditions

$$u(0,x) = \varphi_0(x),\ \partial_t u(0,x) = \varphi_1(x),\ \partial_t^2 u(0,x) = \varphi_2(x),\ \partial_t^3 u(0,x) = \varphi_3(x),\ x \in [0,\infty), \tag{7}$$

and the boundary conditions

$$u(t,0) = \mu_0(t),\ \partial_x^2 u(t,0) = \mu_1(t),\ t \in [0,\infty), \tag{8}$$

where φ_0, φ_1, φ_2, φ_3, μ_0, and μ_1 are the functions defined on the half-line $[0,\infty)$.

As noted above, equations of the form (6) are used for modeling Timoshenko beams [3] in the nonstrictly hyperbolic case, i.e., when the equality $E = \kappa G$ holds, where E is the elastic modulus of the beam material, G is the shear modulus of the beam material, and κ is the Timoshenko shear coefficient. The homogeneous boundary conditions of the form $u(t,0) = \partial_x^2 u(t,0) = 0$ correspond to a simply supported beam, the parameter $a = \sqrt{E\rho^{-1}}$, and the function f can be defined by the formula $\mathcal{F}[u](t,x) := (\kappa AGJ^{-1}m^{-1} + m^{-1}\partial_t^2 - EIJ^{-1}m^{-1})q(t,x) - \kappa AGJ^{-1}\partial_t^2 u(t,x)$, where A is the cross-sectional area of the beam, I is the second moment of cross-sectional area, $q(t,x)$ is a distributed load (force per unit length), $m := \rho A$, and $J := \rho I$.

2. Integro-Differential Equation

The domain Q is divided by the characteristic $x - at = 0$ into two subdomains $Q^{(j)} = \{(t,x) \in Q\colon (-1)^j(at - x) > 0\}$, $j = 1, 2$. In the closure $\overline{Q^{(j)}}$ of each of the subdomains $Q^{(j)}$, the integro-differential equations considered are

$$\begin{aligned}
u^{(1)}(t,x) = & \int\limits_{x-at}^{x+at} \frac{6a^2\varphi_1(z) + 2a^2t\varphi_2(z) + \left(a^2t^2 - (x-z)^2\right)\varphi_3(z)}{8a^3}\,dz + \\
& + \frac{\varphi_0(x-at) + \varphi_0(x+at)}{2} - \frac{t\left(\varphi_1(x-at) + \varphi_1(x+at)\right)}{4} + \\
& + \frac{at\left(D\varphi_0(x-at) - D\varphi_0(x+at)\right)}{4} + \\
& + \int\limits_0^t d\tau \int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} \frac{\left(a^2(t-\tau)^2 - (x-z)^2\right)\mathcal{F}\left[u^{(1)}\right](\tau,z)}{8a^3}\,dz, \quad (t,x) \in \overline{Q^{(1)}},
\end{aligned} \tag{9}$$

$$\begin{aligned}
u^{(2)}(t,x) = & \int\limits_{0}^{x+at} \frac{6a^2\varphi_1(z) + 2a^2t\varphi_2(z) + \left(a^2t^2 - (x-z)^2\right)\varphi_3(z)}{8a^3}\,dz + \\
& + \int\limits_{at-x}^{0} \frac{6a^2\varphi_1(z) + 2a^2t\varphi_2(z) + \left(a^2t^2 - (x+z)^2\right)\varphi_3(z)}{8a^3}\,dz + \\
& + \frac{x}{2}\int\limits_0^{x-at} \mu_1\Big(-\frac{z}{a}\Big)\,dz + \mu_0\Big(t - \frac{x}{a}\Big) + \frac{\varphi_0(x+at) - \varphi_0(at-x)}{2} - \frac{x\varphi_1(0)}{2a} + \\
& + \frac{t\left(\varphi_1(at-x) - \varphi_1(x+at)\right)}{4} - \frac{xD\mu_0(0)}{2a} + \frac{xD\mu_0\Big(t - \dfrac{x}{a}\Big)}{2a} + \\
& + \frac{at\left(D\varphi_0(at-x) - D\varphi_0(x+at)\right)}{4} + \\
& + \int\limits_{t-x/a}^{t} d\tau \int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} \frac{\left(a^2(t-\tau)^2 - (x-z)^2\right)\mathcal{F}\left[u^{(1)}\right](\tau,z)}{8a^3}\,dz + \\
& + \int\limits_0^{t-x/a} d\tau \int\limits_0^{x+a(t-\tau)} \frac{\left(a^2(t-\tau)^2 - (x-z)^2\right)\mathcal{F}\left[u^{(2)}\right](\tau,z)}{8a^3}\,dz + \\
& + \int\limits_0^{t-x/a} d\tau \int\limits_0^{a(t-\tau)-x} \frac{\left((x+z)^2 - a^2(t-\tau)^2\right)\mathcal{F}\left[u^{(2)}\right](\tau,z)}{8a^3}\,dz, \quad (t,x) \in \overline{Q^{(2)}},
\end{aligned} \tag{10}$$

where D is the ordinary derivative operator.

On the closure $\overline{Q}$ of the domain Q, a function u is defined as the one coinciding with the solution $u^{(j)}$ of the integral equations (9) and (10)

$$u(t,x) = u^{(j)}(t,x), \quad (t,x) \in \overline{Q^{(j)}}, \quad j = 1,2, \tag{11}$$

on the closure $\overline{Q^{(j)}}$ of the domain $Q^{(j)}$.

Theorem 1. *Let the conditions* $\varphi_0 \in C^5([0,\infty))$, $\varphi_1 \in C^4([0,\infty))$, $\varphi_2 \in C^3([0,\infty))$, $\varphi_3 \in C^2([0,\infty))$, $\mu_0 \in C^5([0,\infty))$, $\mu_1 \in C^3([0,\infty))$, $f \in C^2\big(\overline{Q} \times \mathbb{R}^6\big)$ *hold. The function* u *belongs to the class* $C^4\big(\overline{Q}\big)$ *and satisfies Eq.* (6), *the initial conditions* (7), *and the boundary conditions* (8) *if and only if, for each* $j = 1,2$, *it is a solution of Eqs.* (9) *and* (10) *in the space* $C^2\big(\overline{Q^{(j)}}\big)$, *subject to the following matching conditions:*

$$D\mu_0(0) = \varphi_1(0), \quad \mu_1(0) = D^2\varphi_0(0), \tag{12}$$

$$D^2\mu_0(0) = \varphi_2(0), \quad D\mu_1(0) = D^2\varphi_1(0), \tag{13}$$

$$D^3\mu_0(0) = \varphi_3(0), \quad D^2\mu_1(0) = D^2\varphi_2(0), \tag{14}$$

$$\begin{gathered} D^4\mu_0(0) = 2a^2D^2\varphi_2(0) - a^4D^4\varphi_0(0) + \\ + f\big(0,0,\varphi_0(0),\varphi_1(0),D\varphi_0(0),\varphi_2(0),D\varphi_1(0),D^2\varphi_0(0)\big), \end{gathered} \tag{15}$$

$$\begin{gathered} D^5\mu_0(0) = a^2D^2\varphi_3(0) + a^2D^3\mu_1(0) - a^4D^4\varphi_1(0) + D^2\varphi_1(0) \times \\ \times \partial_{u_{xx}} f\big(0,0,\varphi_0(0),\varphi_1(0),D\varphi_0(0),\varphi_2(0),D\varphi_1(0),u_{xx} = D^2\varphi_0(0)\big) + \\ + D\varphi_2(0)\partial_{u_{tx}} f\big(0,0,\varphi_0(0),\varphi_1(0),D\varphi_0(0),\varphi_2(0),u_{tx} = D\varphi_1(0),D^2\varphi_0(0)\big) + \\ + \varphi_3(0)\partial_{u_{tt}} f\big(0,0\varphi_0(0),\varphi_1(0),D\varphi_0(0),u_{tt} = \varphi_2(0),D\varphi_1(0),D^2\varphi_0(0)\big) + \\ + D\varphi_1(0)\partial_{u_x} f\big(0,0,\varphi_0(0),\varphi_1(0),u_x = D\varphi_0(0),\varphi_2(0),D\varphi_1(0),D^2\varphi_0(0)\big) + \\ + \varphi_2(0)\partial_{u_t} f\big(0,0,\varphi_0(0),u_t = \varphi_1(0),D\varphi_0(0),\varphi_2(0),D\varphi_1(0),D^2\varphi_0(0)\big) + \\ + \varphi_1(0)\partial_u f\big(0,0,u = \varphi_0(0),\varphi_1(0),D\varphi_0(0),\varphi_2(0),D\varphi_1(0),D^2\varphi_0(0)\big) + \\ + \partial_t f\big(t = 0,0,\varphi_0(0),\varphi_1(0),D\varphi_0(0),\varphi_2(0),D\varphi_1(0),D^2\varphi_0(0)\big). \end{gathered} \tag{16}$$

Proof. 1. Let the function $u \in C^4\big(\overline{Q}\big)$ satisfy Eq. (6) in $\overline{Q}$, the initial conditions (7), and the boundary conditions (8) everywhere. Under a linear nondegenerate change of the independent variables $\xi = x - at$, $\eta = x + at$ and with $u(t,x)$ expressed as $v(\xi,\eta)$, the differential equation is transformed into

$$\partial_\xi^2\partial_\eta^2 v(\xi,\eta) = \frac{1}{16a^4}\mathcal{F}[u]\left(\frac{\eta-\xi}{2a}, \frac{\eta+\xi}{2}\right).$$

Integrating it four times yields the equation

$$\begin{gathered} v(\xi,\eta) = f_1(\xi) + \eta f_2(\xi) + f_3(\eta) + \xi f_4(\eta) + \\ + \frac{1}{16a^4}\int\limits_0^\xi dy \int\limits_{|\xi|}^\eta (\xi - y)(\eta - z)\mathcal{F}[u]\left(\frac{z-y}{2a}, \frac{z+y}{2}\right) dz. \end{gathered}$$

Returning to the original variables t and x, we obtain

$$\begin{gathered} u(t,x) = f_1(x-at) + (x+at)f_2(x-at) + f_3(x+at) + (x-at)f_4(x+at) + \\ + \frac{1}{16a^4}\int\limits_0^{x-at} dy \int\limits_{|x-at|}^{x+at} (x-at-y)(x+at-z)\mathcal{F}[u]\left(\frac{z-y}{2a}, \frac{z+y}{2}\right) dz. \end{gathered} \tag{17}$$

By introducing functions g_1, g_2, g_3, and g_4

$$f_1(z) = g_1(z) - \frac{zg_2(z)}{2a}, \;\; f_2(z) = \frac{g_2(z)}{2a}, \;\; f_3(z) = g_3(z) + \frac{zg_4(z)}{2a}, \;\; f_4(z) = -\frac{g_4(z)}{2a},$$

we can rewrite (17) as

$$u(t,x) = g_1(x-at) + tg_2(x-at) + g_3(x+at) + tg_4(x+at) + \\ + \frac{1}{16a^4} \int_0^{x-at} dy \int_{|x-at|}^{x+at} (x-at-y)(x+at-z)\mathcal{F}[u]\left(\frac{z-y}{2a}, \frac{z+y}{2}\right) dz. \quad (18)$$

Note that the functions g_1, g_2, g_3, and g_4 in the representation (18) should be determined by the initial conditions (7) and the boundary conditions (8). Substituting (18) into (7), we obtain the following system

$$g_1(x) + g_3(x) = \varphi_0(x), \;\; x \in [0,\infty), \quad (19)$$

$$g_2(x) + g_4(x) - aDg_1(x) + aDg_3(x) = \varphi_1(x), \;\; x \in [0,\infty), \quad (20)$$

$$\int_0^x \frac{(x-y)\mathcal{G}(x,y)}{4a^2}\, dy - 2aDg_2(x) + \\ + 2aDg_4(x) + a^2D^2g_1(x) + a^2D^2g_3(x) = \varphi_2(x), \;\; x \in [0,\infty), \quad (21)$$

$$\int_0^x -\frac{(x-y)\partial_x\mathcal{G}(x,y) + 3\mathcal{G}(x,y)}{4a}\, dy + \\ + a^2\left(aD^3g_3(x) - aD^3g_1(x) + 3D^2g_2(x) + 3D^2g_4(x)\right) = \varphi_3(x), \;\; x \in [0,\infty), \quad (22)$$

where $G(y,z)$ is denoted as

$$\mathcal{G}(y,z) = \mathcal{F}[u]\left(\frac{z-y}{2a}, \frac{z+y}{2}\right).$$

From (19) and (20), we have

$$g_1(x) = \varphi_0(x) - g_3(x), \;\; g_2(x) = \varphi_1(x) - g_4(x) + aDg_1(x) - aDg_3(x), \;\; x \in [0,\infty). \quad (23)$$

Substituting (23) into (21) and (22), we get two ordinary differential equations

$$D^3g_3(x) = \int_0^x -\frac{(x-y)\partial_x\mathcal{G}(y,x) + 3\mathcal{G}(y,x)}{16a^4}\, dy - \\ - \frac{\varphi_3(x)}{4a^3} + \frac{3D^2\varphi_1(x)}{4a} + \frac{D^3\varphi_0(x)}{2}, \quad x \in [0,\infty), \quad (24)$$

$$Dg_4(x) = \int_0^x -\frac{(x-y)\mathcal{G}(y,x)}{16a^3}\, dy - \\ - aD^2g_3(x) + \frac{aD^2\varphi_0(x)}{4} + \frac{\varphi_2(x)}{4a} + \frac{D\varphi_1(x)}{2}, \quad x \in [0,\infty). \quad (25)$$

Let us integrate Eqs. (24) and (25)

$$g_3(x) = C_1 + C_2 x + C_3 x^2 + \frac{1}{2}\int\limits_0^x (x-\xi)^2 d\xi \int\limits_0^\xi -\frac{(\xi-y)\partial_\xi \mathcal{G}(y,\xi) + 3\mathcal{G}(y,\xi)}{16a^4}\,dy + \\ + \frac{\varphi_0(x)}{2} - \frac{1}{8a^3}\int\limits_0^x (x-\xi)^2 \varphi_3(\xi)\,d\xi + \frac{3}{4a}\int\limits_0^x \varphi_1(\xi)\,d\xi, \quad x \in [0,\infty), \tag{26}$$

$$g_4(x) = C_4 - aDg_3(x) + \frac{1}{4a}\int\limits_0^x \varphi_2(\xi)\,d\xi + \frac{aD\varphi_0(x)}{4} + \\ + \int\limits_0^x d\xi \int\limits_0^\xi -\frac{(\xi-y)\mathcal{G}(y,\xi)}{16a^3}\,dy + \frac{\varphi_1(x)}{2}, \quad x \in [0,\infty), \tag{27}$$

where C_1, C_2, C_3, and C_4 are the integration constants. Substituting (23), (26), and (27) into (18) results in

$$u(t,x) = \int\limits_{x-at}^{x+at} \frac{6a^2\varphi_1(z) + 2a^2 t\varphi_2(z) + \left(a^2t^2 - (x-z)^2\right)\varphi_3(z)}{8a^3}\,dz + \\ + \frac{\varphi_0(x-at) + \varphi_0(x+at)}{2} - \frac{t\left(\varphi_1(x-at) + \varphi_1(x+at)\right)}{4} + \\ + \frac{at\left(D\varphi_0(x-at) - D\varphi_0(x+at)\right)}{4} + \\ + \frac{1}{16a^4}\int\limits_0^{x-at} dy \int\limits_{x-at}^{x+at} (x-at-y)(x+at-z)\mathcal{F}[u]\left(\frac{z-y}{2a}, \frac{z+y}{2}\right) dz + \\ + \int\limits_0^{x-at} dz \int\limits_0^z \frac{\mathcal{G}(y,z)\left(-3a^2t^2 + 2at(z-y) + 3(x-z)^2\right) - (y-z)\partial_z\mathcal{G}(y,z)\left((x-z)^2 - a^2t^2\right)}{32a^4}\,dy + \\ + \int\limits_0^{x+at} dz \int\limits_0^z \frac{(y-z)\partial_z\mathcal{G}(y,z)\left((x-z)^2 - a^2t^2\right) + \mathcal{G}(y,z)\left(3a^2t^2 + 2at(y-z) - 3(x-z)^2\right)}{32a^4}\,dy, \\ (t,x) \in \overline{Q^{(1)}}. \tag{28}$$

To simplify the expression (28), we integrate by parts, i.e.,

$$\int\limits_0^{x+at} dz \int\limits_0^z \frac{(y-z)\left((x-z)^2 - a^2t^2\right)\partial_z\mathcal{G}(y,z)}{32a^4}\,dy = \\ = \int\limits_0^{x+at} \left((x-z)^2 - a^2t^2\right) dz \int\limits_0^z \frac{(y-z)\partial_z\mathcal{G}(y,z)}{32a^4}\,dy = \\ = \int\limits_0^{x+at} \left((x-z)^2 - a^2t^2\right) dz \int\limits_0^z \frac{(y-z)\partial_z\mathcal{G}(y,z) - \mathcal{G}(y,z) + \mathcal{G}(y,z)}{32a^4}\,dy =$$

$$= \int\limits_0^{x+at} \big((x-z)^2 - a^2t^2\big)\,dz \int\limits_0^z \frac{(y-z)\partial_z\mathcal{G}(y,z) - \mathcal{G}(y,z)}{32a^4}\,dy +$$

$$+ \int\limits_0^{x+at} \big((x-z)^2 - a^2t^2\big)\,dz \int\limits_0^z \frac{\mathcal{G}(y,z)}{32a^4}\,dy =$$

$$= \left[\begin{matrix} U = (x-z)^2 - a^2t^2, & dU = 2(z-x)\,dz \\ dV = \Big[\int\limits_0^z \frac{(y-z)\partial_z\mathcal{G}(y,z) - \mathcal{G}(y,z)}{32a^4}\,dy\Big]dz, & V = \int\limits_0^z \frac{(y-z)\mathcal{G}(y,z)}{32a^4}\,dy \end{matrix}\right] =$$

$$= UV\Big|_{z=0}^{z=x+at} - \int\limits_0^{x+at} V\,dU + \int\limits_0^{x+at} \big((x-z)^2 - a^2t^2\big)\,dz \int\limits_0^z \frac{\mathcal{G}(y,z)}{32a^4}\,dy =$$

$$= \int\limits_0^{at+x} dz \int\limits_0^z \frac{(x-z)(y-z)\mathcal{G}(y,z)}{16a^4}\,dy + \int\limits_0^{x+at} \big((x-z)^2 - a^2t^2\big)\,dz \int\limits_0^z \frac{\mathcal{G}(y,z)}{32a^4}\,dy. \quad (29)$$

The resulting expression is

$$\int\limits_0^{x+at} dz \int\limits_0^z \frac{(y-z)\partial_z\mathcal{G}(y,z)\big((x-z)^2 - a^2t^2\big) + \mathcal{G}(y,z)\big(3a^2t^2 + 2at(y-z) - 3(x-z)^2\big)}{32a^4}\,dy =$$

$$= \int\limits_0^{x+at} dz \int\limits_0^z \frac{(at-x+y)(at+x-z)\mathcal{G}(y,z)}{16a^4}\,dy.$$

Similarly, the following calculation is performed:

$$\int\limits_0^{x-at} dz \int\limits_0^z \frac{\mathcal{G}(y,z)\big(-3a^2t^2 + 2at(z-y) + 3(x-z)^2\big) - (y-z)\partial_z\mathcal{G}(y,z)\big((x-z)^2 - a^2t^2\big)}{32a^4}\,dy =$$

$$= \int\limits_{x-at}^{0} dz \int\limits_0^z \frac{(at-x+y)(at+x-z)\mathcal{G}(y,z)}{16a^4}\,dy.$$

Thus, there is an equation

$$u(t,x) = \int\limits_{x-at}^{x+at} \frac{6a^2\varphi_1(z) + 2a^2t\varphi_2(z) + \big(a^2t^2 - (x-z)^2\big)\,\varphi_3(z)}{8a^3}\,dz +$$

$$+ \frac{\varphi_0(x-at) + \varphi_0(x+at)}{2} - \frac{t\,(\varphi_1(x-at) + \varphi_1(x+at))}{4} +$$

$$+ \frac{at\,(D\varphi_0(x-at) - D\varphi_0(x+at))}{4} +$$

$$+ \frac{1}{16a^4} \int\limits_0^{x-at} dy \int\limits_{x-at}^{x+at} (x-at-y)(x+at-z)\mathcal{F}[u]\Big(\frac{z-y}{2a}, \frac{z+y}{2}\Big)\,dz -$$

$$-\frac{1}{16a^4}\int\limits_{x-at}^{x+at} dz \int\limits_0^z (x-at-y)(x+at-z)\mathcal{F}[u]\left(\frac{z-y}{2a}, \frac{z+y}{2}\right) dy, \quad (t,x) \in \overline{Q^{(1)}}. \tag{30}$$

Then, changing the variables $\tau = (z-y)/(2a)$, $\xi = (z+y)/(2)$ in the double integral in the formula (30), we arrive at Eq. (9).

To define the functions g_1 and g_2 for the negative values of the argument, the boundary conditions (8) are used. Substituting the expression (18) yields the equations

$$g_1(-at) + tg_2(-at) + g_3(at) + tg_4(at) = \mu_0(t), \quad t \in [0,\infty), \tag{31}$$

$$\int\limits_0^{-at} \frac{(-at-y)\mathcal{G}(y,a\ t)}{4a^4}\, dy + D^2 g_1(-at) + tD^2 g_2(-at) + \tag{32}$$

$$+ D^2 g_3(at) + tD^2 g_4(at) = \mu_1(t), \quad t \in [0,\infty). \tag{33}$$

From (31), we have

$$g_1(z) = \frac{-ag_3(-z) + a\mu_0\left(-\frac{z}{a}\right) + zg_2(z) + zg_4(-z)}{a}, \quad z \in (-\infty, 0]. \tag{34}$$

Substituting (34) into (33) leads to a first-order ordinary differential equation

$$Dg_2(z) = \int\limits_0^z -\frac{(z-y)\mathcal{G}(y,-z)}{8a^3}\, dy - \frac{D^2\mu_0\left(-\frac{z}{a}\right)}{2a} + \frac{a\mu_1\left(-\frac{z}{a}\right)}{2} + Dg_4(-z),$$

$$z \in (-\infty, 0]. \tag{35}$$

Integrating (35), we get

$$g_2(z) = g_2(0) + \int\limits_0^z d\xi \int\limits_0^\xi -\frac{(\xi-y)\mathcal{G}(y,-\xi)}{8a^3}\, dy + \frac{a}{2}\int\limits_0^z \mu_1\left(-\frac{\xi}{a}\right) d\xi + g_4(0) -$$

$$- g_4(-z) + \frac{D\mu_0\left(-\frac{z}{a}\right)}{2a^2} - \frac{D\mu_0(0)}{2a^2}, \quad z \in (-\infty, 0], \tag{36}$$

where the values $g_2(0)$, $g_4(0)$, and $g_4(-z)$ can be calculated by the formulas (23) and (27). Then, using the representations (23), (26), (27), (34), and (36), we substitute the functions g_1, g_2, g_3, and g_4 into the formula (17) for $(t,x) \in \overline{Q^{(2)}}$, integrate by parts as in (29), and get Eq. (10).

The continuity conditions for the function u and its partial derivatives up to and including the fourth order, i.e.,

$$\partial_t^k \partial_x^p u^{(1)}(t, x = at) = \partial_t^k \partial_x^p u^{(2)}(t, x = at), \quad 0 \leqslant k + p \leqslant 4, \tag{37}$$

are also satisfied, where k and p are nonnegative integers. It turns out that the equalities (37) entail the matching conditions (12)–(16), which can be verified directly using the algorithm outlined in [16]. Note that, in this case, the conditions (12)–(16) cannot be strictly justified by differentiating the initial and boundary conditions, as was done, for example, in [17].

2. Assume that the representations (9)–(11) hold for the function u, which belongs to the classes $C^2\big(\overline{Q^{(1)}}\big)$ and $C^2\big(\overline{Q^{(2)}}\big)$, and the conditions (12)–(16) are satisfied. Then, by virtue of the smoothness conditions $\varphi_0 \in C^5([0,\infty))$, $\varphi_1 \in C^4([0,\infty))$, $\varphi_2 \in C^3([0,\infty))$, $\varphi_3 \in C^2([0,\infty))$, $\mu_0 \in C^5([0,\infty))$, $\mu_1 \in C^3([0,\infty))$, $f \in C^2\big(\overline{Q} \times \mathbb{R}^6\big)$, similar to [18], it follows that the function u belongs to the classes $C^4\big(\overline{Q^{(1)}}\big)$ and $C^4\big(\overline{Q^{(2)}}\big)$. We substitute the representations (9)–(11) into Eq. (6) and verify that the function u satisfies this equation in $\overline{Q^{(1)}}$ and $\overline{Q^{(2)}}$. In this case, for the function u to belong to the class $C^4\big(\overline{Q}\big)$, it is sufficient that the values of the functions $u^{(2)}$ and $u^{(2)}$ and the values of their derivatives up to and including the fourth order coincide with each other on the characteristic $x = at$, i.e., that the equalities (37) hold. The latter is equivalent to the validity of the conditions (12)–(16), as can be easily derived by following the argument in the reverse order to that in item 1 of the proof, based on the representations (9)–(11). □

Theorem 2. *Let the conditions* $\varphi_0 \in C^3([0,\infty))$, $\varphi_1 \in C^2([0,\infty))$, $\varphi_2 \in C^1([0,\infty))$, $\varphi_3 \in C([0,\infty))$, $\mu_0 \in C^3([0,\infty))$, $\mu_1 \in C^1([0,\infty))$, $f \in C\big(\overline{Q} \times \mathbb{R}^6\big)$ *hold and the function* f *satisfy the Lipschitz condition with* $L \in C(\overline{Q})$ *in the last six variables, i.e., there exists the function* $L \in C(\overline{Q})$ *such that*

$$\big|f(t,x,u_1,u_2,u_3,u_4,u_5,u_6) - f(t,x,z_1,z_2,z_3,z_4,z_5,z_6)\big| \leqslant L(t,x)\sum_{i=1}^{6}|u_i - z_i|.$$

Then, there exist unique solutions of Eqs. (9) *and* (10) *in the spaces* $C^2\big(\overline{Q^{(1)}}\big)$ *and* $C^2\big(\overline{Q^{(2)}}\big)$, *respectively, and these solutions continuously depend on the initial data.*

Proof. To be definite, consider Eq. (9) for the function $u^{(1)}$, which can be solved by the parameter continuation method [19, 20]. Set

$$\begin{aligned} v(t,x) = &\int\limits_{x-at}^{x+at} \frac{6a^2\varphi_1(z) + 2a^2t\varphi_2(z) + \big(a^2t^2 - (x-z)^2\big)\,\varphi_3(z)}{8a^3}\,dz + \\ &+ \frac{\varphi_0(x-at)+\varphi_0(x+at)}{2} - \frac{t\,(\varphi_1(x-at)+\varphi_1(x+at))}{4} + \\ &+ \frac{at\,(D\varphi_0(x-at) - D\varphi_0(x+at))}{4}, \end{aligned}$$

$$K[u](t,x) = \int\limits_0^t d\tau \int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} \frac{\big(a^2(t-\tau)^2 - (x-z)^2\big)\,\mathcal{F}[u](\tau,z)}{8a^3}\,dz.$$

Rewrite Eq. (9) in the operator form

$$u^{(1)}(t,x) = K[u^{(1)}](t,x) + v(t,x), \quad (t,x) \in \overline{Q^{(1)}}. \tag{38}$$

Let us also introduce the following family of equations with the parameter $\varepsilon \in [0,1]$:

$$u^{(1)}_\varepsilon(t,x) - \varepsilon(K[u^{(1)}_\varepsilon] - K[0])(t,x) = w(t,x), \quad (t,x) \in \overline{Q^{(1)}}, \tag{39}$$

where $w(t,x) = v(t,x) + K[0](t,x)$. It is clear that any solution $u^{(1)}_\varepsilon(t,x)$ of Eq. (39) with $\varepsilon = 1$ is also a solution of Eq. (38), and vice versa. Hence, the task reduces to solving Eq. (39) when $\varepsilon = 1$.

Let us introduce the set $\Omega_m = \{(t,x) \mid (t,x) \in \overline{Q^{(1)}} \wedge x + at \leqslant m\}$, $m \in \mathbb{N}$. Due to the smoothness conditions $\varphi_0 \in C^3([0,\infty))$, $\varphi_1 \in C^2([0,\infty))$, $\varphi_2 \in C^1([0,\infty))$, $\varphi_3 \in C([0,\infty))$, $\mu_0 \in C^3([0,\infty))$, $\mu_1 \in C^1([0,\infty))$, $f \in C\big(\overline{Q} \times \mathbb{R}^6\big)$, as in [18], we conclude that $K[g] \in C^2(\Omega_m)$, assuming that, for example, $g \in C^2(\Omega_m)$. It implies that the operator K maps from the space $C^2(\Omega_m)$ to the space $C^2(\Omega_m)$. Let us show that the operator $K\colon C^2(\Omega_m) \mapsto C^2(\Omega_m)$ is Lipschitz-continuous. We have

$$
\begin{gathered}
|K[u_1](t,x) - K[u_2](t,x)| = \Bigg| \int\limits_0^t d\tau \int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} \frac{\big(a^2(t-\tau)^2 - (x-z)^2\big)\,\mathcal{F}[u_1](\tau,z)}{8a^3}\, dz - \\
- \int\limits_0^t d\tau \int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} \frac{\big(a^2(t-\tau)^2 - (x-z)^2\big)\,\mathcal{F}[u_2](\tau,z)}{8a^3}\, dz \Bigg| \leqslant \\
\leqslant \Bigg| \int\limits_0^t d\tau \int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} \frac{\big(a^2(t-\tau)^2 - (x-z)^2\big)\,(\mathcal{F}[u_1](\tau,z) - \mathcal{F}[u_2](\tau,z))}{8a^3}\, dz \Bigg| \leqslant \\
\leqslant \alpha \int\limits_0^t d\tau \int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} |L(t,x)|\,(|u_1 - u_2| + |\partial_t u_1 - \partial_t u_2| + |\partial_x u_1 - \partial_x u_2| + \\
+ |\partial_t^2 u_1 - \partial_t^2 u_2| + |\partial_t\partial_x u_1 - \partial_t\partial_x u_2| + |\partial_x^2 u_1 - \partial_x^2 u_2|)(t,x)\, dz \leqslant \\
\leqslant \alpha a \|L\|_{C(\Omega_m)} \|u_1 - u_2\|_{C^2(\Omega_m)} t^2 \leqslant \alpha m \|L\|_{C(\Omega_m)} \|u_1 - u_2\|_{C^2(\Omega_m)}, \\
(t,x) \in \Omega_m,\ u_1 \in C^2(\Omega_m),\ u_2 \in C^2(\Omega_m), \quad (40)
\end{gathered}
$$

where

$$
\alpha = \max_{(t,x)\in\Omega_m} \left| \frac{a^2(t-\tau)^2 - (x-z)^2}{8a^3} \right|.
$$

Proceeding to (40), we arrive at the estimate

$$
\begin{gathered}
|\partial_t^p \partial_x^k K[u_1](t,x) - \partial_t^p \partial_x^k K[u_2](t,x)| \leqslant \alpha_{p,k} \|u_1 - u_2\|_{C^2(\Omega_m)},\ (t,x) \in \Omega_m, \\
0 \leqslant p + k \leqslant 2,\ u_1 \in C^2(\Omega_m),\ u_2 \in C^2(\Omega_m), \quad (41)
\end{gathered}
$$

where k and p are nonnegative integers, and $\alpha_{p,k}$ is a constant determined by the function L, the number a, and the set Ω_m. It follows from (41) that

$$
\|K[u_1] - K[u_2]\|_{C^2(\Omega_m)} \leqslant \beta \|u_1 - u_2\|_{C^2(\Omega_m)},\ u_1 \in C^2(\Omega_m),\ u_2 \in C^2(\Omega_m), \tag{42}
$$

where $\beta = \alpha_{0,0} + \alpha_{1,0} + \alpha_{0,1} + \alpha_{2,0} + \alpha_{1,1} + \alpha_{0,2}$. The inequality (42) implies that the operator $K\colon C^2(\Omega_m) \mapsto C^2(\Omega_m)$ is Lipschitz-continuous.

Consider the operator K_ε defined by the formula

$$
K_\varepsilon[u] = u - \varepsilon(K[u] - K[0]).
$$

Since the operator $K\colon C^2(\Omega_m) \mapsto C^2(\Omega_m)$ is Lipschitz-continuous, the operator $K_\varepsilon\colon C^2(\Omega_m) \mapsto C^2(\Omega_m)$ retains this property.

Let us prove that the operator $K_\varepsilon\colon C^2(\Omega_m) \mapsto C^2(\Omega_m)$ is coercive. To achieve this, it suffices to derive an a priori estimate of the form

$$
\big\|u_\varepsilon^{(1)}\big\|_{C^2(\Omega_m)} \leqslant C \|w\|_{C^2(\Omega_m)}, \tag{43}
$$

for the solution $u_\varepsilon^{(1)}$ of Eq. (39), where C is some constant that does not depend on the function $u_\varepsilon^{(1)}$ and the number ε. We have

$$|u_\varepsilon^{(1)}(t,x)| = |w(t,x) + \varepsilon(J[u_\varepsilon^{(1)}](t,x) - K[0](t,x))| \leqslant \|w\|_{C(\Omega_m)} + \\ + \left|\varepsilon \int_0^t d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} \frac{\left(a^2(t-\tau)^2 - (x-z)^2\right)\left(\mathcal{F}\big[u^{(1)}\big](\tau,z) - \mathcal{F}[0](\tau,z)\right)}{8a^3}\, dz\right| \leqslant \\ \leqslant \|w\|_{C(\Omega_m)} + \alpha \int_0^t d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} U(\tau,z)\, dz, \ \ (t,x) \in \Omega_m, \quad (44)$$

where

$$U(t,x) = |u^{(1)}(t,x)| + |\partial_t u^{(1)}(t,x)| + |\partial_x u^{(1)}(t,x)| + \\ + |\partial_t^2 u^{(1)}(t,x)| + |\partial_t\partial_x u^{(1)}(t,x)| + |\partial_x^2 u^{(1)}(t,x)|. \quad (45)$$

Similarly, we get

$$|\partial_t u_\varepsilon^{(1)}(t,x)| \leqslant \|\partial_t w\|_{C(\Omega_m)} + \gamma_{1,0} \int_0^t d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} U(\tau,z)\, dz, \ \ (t,x) \in \Omega_m, \quad (46)$$

$$|\partial_x u_\varepsilon^{(1)}(t,x)| \leqslant \|\partial_x w\|_{C(\Omega_m)} + \gamma_{0,1} \int_0^t d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} U(\tau,z)\, dz, \ \ (t,x) \in \Omega_m, \quad (47)$$

$$|\partial_t^2 u_\varepsilon^{(1)}(t,x)| \leqslant \\ \leqslant \|\partial_t^2 w\|_{C(\Omega_m)} + \lambda_1 \int_0^t U(\tau, x - a(t-\tau))\, d\tau + \psi_1 \int_0^t U(\tau, x + a(t-\tau))\, d\tau + \\ + \gamma_{2,0} \int_0^t d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} U(\tau,z)\, dz, \ \ (t,x) \in \Omega_m, \quad (48)$$

$$|\partial_t\partial_x u_\varepsilon^{(1)}(t,x)| \leqslant \\ \leqslant \|\partial_t\partial_x w\|_{C(\Omega_m)} + \lambda_2 \int_0^t U(\tau, x - a(t-\tau))\, d\tau + \psi_2 \int_0^t U(\tau, x + a(t-\tau))\, d\tau + \\ + \gamma_{1,1} \int_0^t d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} U(\tau,z)\, dz, \ \ (t,x) \in \Omega_m, \quad (49)$$

$$|\partial_x^2 u_\varepsilon^{(1)}(t,x)| \leqslant \\ \leqslant \|\partial_x^2 w\|_{C(\Omega_m)} + \lambda_3 \int_0^t U(\tau, x - a(t-\tau))\, d\tau + \psi_3 \int_0^t U(\tau, x + a(t-\tau))\, d\tau + \\ + \gamma_{0,2} \int_0^t d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} U(\tau,z)\, dz, \ \ (t,x) \in \Omega_m, \quad (50)$$

where $\gamma_{1,0}$, $\gamma_{0,1}$, $\gamma_{2,0}$, $\gamma_{1,1}$, $\gamma_{0,2}$, λ_i $(i=1,2,3)$, and ψ_i $(i=1,2,3)$ are the constants, which depend on the function L, the number a, and the set Ω_m.

Summation of the inequalities (44)–(50) yields

$$|U(t,x)| \leqslant \|w\|_{C^2(\Omega_m)} + \lambda \int_0^t U(\tau, x-a(t-\tau))\,d\tau + \psi \int_0^t U(\tau, x+a(t-\tau))\,d\tau + \\ + \gamma \int_0^t d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} U(\tau,z)\,dz, \quad (t,x)\in\Omega_m, \quad (51)$$

where $\lambda = \lambda_1+\lambda_2+\lambda_3$, $\psi=\psi_1+\psi_2+\psi_3$, $\gamma = \alpha+\gamma_{1,0}+\gamma_{0,1}+\gamma_{2,0}+\gamma_{1,1}+\gamma_{0,2}$.

Let us denote $V(s) = U(s, x-a(t-s))$ for fixed $x \in \{x \,|\, \exists t : (t,x)\in\Omega_m\}$. Then, we have

$$|V(t)| \leqslant \|w\|_{C^2(\Omega_m)} + \lambda \int_0^t V(\tau)\,d\tau + \psi \int_0^t U(\tau, x+a(t-\tau))\,d\tau + \\ + \gamma \int_0^t d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} U(\tau,z)\,dz, \quad (t,x)\in\Omega_m,$$

Applying the Grönwall lemma to the preceding inequality, we obtain

$$|V(t)| \leqslant \Bigg(\|w\|_{C^2(\Omega_m)} + \psi \int_0^t U(\tau, x+a(t-\tau))\,d\tau + \\ + \gamma \int_0^t d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} U(\tau,z)\,dz \Bigg) \exp(\lambda t), \quad (t,x)\in\Omega_m.$$

Using this technique iteratively, we get

$$|U(t,x)| \leqslant \Bigg(\|w\|_{C^2(\Omega_m)} + \gamma \int_0^t d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} U(\tau,z)\,dz \Bigg) \exp\left(\frac{\psi m}{2a} \exp\left(\frac{\lambda m}{2a}\right)\right), \\ (t,x)\in\Omega_m.$$

Applying the multidimensional Grönwall lemma [21] to the preceding inequality, we derive the estimate

$$|U(t,x)| \leqslant \|w\|_{C^2(\Omega_m)} \exp\left(\frac{\psi m}{2a}\exp\left(\frac{\lambda m}{2a}\right)\right) \exp\left(\gamma a t^2 \exp\left(\frac{\psi m}{2a}\exp\left(\frac{\lambda m}{2a}\right)\right)\right), \\ (t,x)\in\Omega_m. \quad (52)$$

The formulas (45) and (52) are actually the a priori estimates of the form (43). Therefore, we have proved that the operator $K_\varepsilon \colon C^2(\Omega_m) \mapsto C^2(\Omega_m)$ is coercive.

Note that the function $B\colon [0,1] \ni \varepsilon \mapsto A_\varepsilon$ is continuous in the seminorm of the space of Lipschitz-continuous operators [19]. It is obvious that $B(0) = A_0$ is continuously invertible, as it corresponds to the identity operator. Considering this, we conclude that the conditions of [19, Theorem 4] hold for the operator-function B. Therefore, Eq. (39) has a unique solution in the space $C^2(\Omega_m)$ for any $\varepsilon \in [0,1]$, and this solution continuously depends on the initial data. Thus, we have solved Eq. (38) in the space $C^2(\Omega_m)$.

To construct the solution u of Eq. (38) in the space $C^2\big(\overline{Q^{(1)}}\big)$, consider the following limit

$$u = \lim_{m\to\infty} u_m, \tag{53}$$

where u_m $(m \in \mathbb{N})$ is the solution of Eq. (38) in the space $C^2(\Omega_m)$. We also assume that the functions $u_m \in C^2\big(\overline{Q^{(1)}}\big)$ are extended in some way outside the set Ω_m.

Let us prove the existence of the limit (53). Consider the functions u_n and u_m, where $n < m$. In this case, $u_m|_{\Omega_n} \equiv u_n$. Otherwise, there would be a contradiction with the uniqueness of the solution of Eq. (38) in the class $C^2(\Omega_n)$. Thus, for any $\varepsilon > 0$, there exists an integer $N(\varepsilon) = m$ such that for any integer $M > N(\varepsilon)$ we have $\|u_N - u_M\|_{C^2(\Omega_m)} < \varepsilon$. This indicates that the sequence (u_m) is fundamental in any seminorm of the form $\|\cdot\|_{C^2(\Omega_k)}$, where $k \in \mathbb{N}$. Since $\bigcup\limits_{m=1}^{\infty} \Omega_m = \overline{Q^{(1)}}$, the topology of the Fréchet space $C^2\big(\overline{Q^{(1)}}\big)$ can be induced by a countable family of seminorms $\|\cdot\|_{C^2(\Omega_k)}$. So, the sequence (u_m) converges in the space $C^2\big(\overline{Q^{(1)}}\big)$.

Next, we can prove that the limit (53) solves Eq. (38). Consider a point $(t_0, x_0) \in \overline{Q^{(1)}}$. There exists a number m such that $(t_0, x_0) \in \Omega_m$. We have $u|_{\Omega_n} \equiv u_m$, where $m > n$. Otherwise, there would be a contradiction with the uniqueness of the solution of Eq. (38) in the class $C^2(\Omega_n)$. Then,

$$u(t_0, x_0) = u_m(t_0, x_0) = K[u_m](t_0, x_0) + w(t, x). \tag{54}$$

Let us pass to the limit as $m \to \infty$ in (54) and obtain

$$\begin{aligned} u(t_0, x_0) = \lim_{m\to\infty} (K[u_m](t_0, x_0) + w(t_0, x_0)) = K\big[\lim_{m\to\infty} u_m\big](t_0, x_0) + w(t_0, x_0) = \\ = K[u](t_0, x_0) + w(t_0, x_0). \end{aligned}$$

Given the arbitrariness of the point $(t_0, x_0) \in \overline{Q^{(1)}}$ and the preceding equality, we conclude that the function u defined by the limit (53) is a solution of Eq. (38) in the class $C^2\big(\overline{Q}\big)$.

Let us prove that the limit (53) is the unique solution of Eq. (38). Assume that Eq. (38) has two solutions u and $\widetilde{u}$ in the space $C^2\big(\overline{Q^{(1)}}\big)$. Then, the functions $u|_{\Omega_m}$ and $\widetilde{u}|_{\Omega_m}$ are the solutions of Eq. (38) in the class $C^2(\Omega_m)$. Therefore, $u|_{\Omega_m} \equiv \widetilde{u}|_{\Omega_m}$. Since $\bigcup\limits_{m=1}^{\infty} \Omega_m = \overline{Q^{(1)}}$, we arrive at the equality $u \equiv \widetilde{u}$. This proves that Eq. (38) has a unique solution in the class $C^2\big(\overline{Q^{(1)}}\big)$.

Therefore, we have constructed a unique solution of Eq. (9) in the class $C^2\big(\overline{Q^{(1)}}\big)$. The existence of a unique solution of Eq. (10) in the class $C^2\big(\overline{Q^{(2)}}\big)$, which continuously depends on the initial data, can be proved in a similar way. □

3. Classical Solution

The following theorem is a consequence of Theorems 1 and 2.

Theorem 3. *Let the conditions* $\varphi_0 \in C^5([0,\infty))$, $\varphi_1 \in C^4([0,\infty))$, $\varphi_2 \in C^3([0,\infty))$, $\varphi_3 \in C^2([0,\infty))$, $\mu_0 \in C^5([0,\infty))$, $\mu_1 \in C^3([0,\infty))$, $f \in C^2\big(\overline{Q} \times \mathbb{R}^6\big)$ *hold and the function* f *satisfy the Lipschitz condition with* $L \in C(\overline{Q})$ *in the last six variables, i.e., there exists the function* $L \in C(\overline{Q})$ *such that*

$$\big|f(t,x,u_1,u_2,u_3,u_4,u_5,u_6) - f(t,x,z_1,z_2,z_3,z_4,z_5,z_6)\big| \leqslant L(t,x)\sum_{i=1}^{6}|u_i - z_i|.$$

Then, the initial-boundary value problem (6)–(8) *has a unique solution* u *in the class* $C^4\big(\overline{Q}\big)$ *if and only if the conditions* (12)–(16) *are satisfied. This solution is determined by the formulas* (9)–(11).

4. Mild Solution

Consider the problem (6)–(8) for the case, where the functions φ_0, φ_1, φ_2, φ_3, μ_0, μ_1, and f are not smooth enough.

Definition 1. We define the function u representable in the form (9)–(11) as a mild solution of the problem (6)–(8).

Remark 1. Any classical solution of the problem (6)–(8) is also a mild solution of this problem.

Remark 2. If the additional smoothness conditions $\varphi_0 \in C^5([0,\infty))$, $\varphi_1 \in C^4([0,\infty))$, $\varphi_2 \in C^3([0,\infty))$, $\varphi_3 \in C^2([0,\infty))$, $\mu_0 \in C^5([0,\infty))$, $\mu_1 \in C^3([0,\infty))$, $f \in C^2\big(\overline{Q} \times \mathbb{R}^6\big)$ and the matching conditions (12)–(16) hold, then the mild solution of problem (6)–(8) is classical.

Let $\widetilde{Q} = \overline{Q} \setminus \{(t,x) \mid x = at\}$.

Theorem 4. *Let the conditions* $\varphi_0 \in C^3([0,\infty))$, $\varphi_1 \in C^2([0,\infty))$, $\varphi_2 \in C^1([0,\infty))$, $\varphi_3 \in C([0,\infty))$, $\mu_0 \in C^3([0,\infty))$, $\mu_1 \in C^1([0,\infty))$, $f \in C\big(\overline{Q} \times \mathbb{R}^6\big)$ *hold and the function* f *satisfy the Lipschitz condition with* $L \in C(\overline{Q})$ *in the last six variables, i.e., there exists the function* $L \in C(\overline{Q})$ *such that*

$$\big|f(t,x,u_1,u_2,u_3,u_4,u_5,u_6) - f(t,x,z_1,z_2,z_3,z_4,z_5,z_6)\big| \leqslant L(t,x)\sum_{i=1}^{6}|u_i - z_i|.$$

Then, the initial-boundary value problem (6)–(8) *has a mild solution* u *in the class* $C^2\big(\widetilde{Q}\big)$.

Proof. The solvability of the integral equations (9) and (10) and the belonging of their solutions to the classes of $C^2\big(\overline{Q^{(1)}}\big)$ and $C^2\big(\overline{Q^{(2)}}\big)$, respectively, follows from Theorem 2. □

If the matching conditions (12)–(16) are partially met, the smoothness of the mild solution can be increased, i.e., the following theorem holds.

Theorem 5. *Let the conditions* $\varphi_0 \in C^3([0,\infty))$, $\varphi_1 \in C^2([0,\infty))$, $\varphi_2 \in C^1([0,\infty))$, $\varphi_3 \in C([0,\infty))$, $\mu_0 \in C^3([0,\infty))$, $\mu_1 \in C^1([0,\infty))$, $f \in C(\overline{Q} \times \mathbb{R}^6)$ *hold and the function* f *satisfy the Lipschitz condition with* $L \in C(\overline{Q})$ *in the last six variables, i.e., there exists the function* $L \in C(\overline{Q})$ *such that*

$$\left| f(t,x,u_1,u_2,u_3,u_4,u_5,u_6) - f(t,x,z_1,z_2,z_3,z_4,z_5,z_6)\right| \leqslant L(t,x)\sum_{i=1}^{6}|u_i - z_i|.$$

Then, the initial-boundary value problem (6)–(8) *has a mild solution* u *in the class* $C^2(\widetilde{Q}) \cap C(\overline{Q})$ *if and only if* $\varphi_0(0) = \mu_0(0)$.

Proof. 1. Let us prove the necessity of the condition $\varphi_0(0)=\mu_0(0)$. If $u \in C(\overline{Q})$, then $u(0,0) = \lim\limits_{t\to 0} u(t,0) = \lim\limits_{x\to 0} u(0,x)$. The representations (9)–(11) imply $\lim\limits_{t\to 0} u(t,0) = \lim\limits_{t\to 0} \mu_0(t) = \mu(0)$ and $\lim\limits_{x\to 0} u(0,x) = \lim\limits_{x\to 0} \varphi_0(x) = \varphi_0(0)$. Hence, $\varphi_0(0)=\mu_0(0)$.

2. Let us prove the sufficiency of the condition $\varphi_0(0) = \mu_0(0)$. According to Theorem 4, there exist a unique mild solution $u \in C^2(\widetilde{Q})$ of the problem (6)–(8). Using the formulas (9)–(11), we compute

$$[(u)^+ - (u)^-](t, x = at) = u^{(1)}(t,at) - u^{(2)}(t,at) = \varphi_0(0) - \mu_0(0), \tag{55}$$

where $(u)^{\pm}(t, x = at) = \lim\limits_{\delta\to 0+} u(t, a\,t \pm \delta)$. Using (55), we conclude that $u \in C(\widetilde{Q})$. □

Conclusions

Sufficient conditions for the existence of a unique classical solution of the initial-boundary value problem in the first quadrant for a nonstrictly hyperbolic mildly quasi-linear biwave equation are established. The results obtained show that the failure to meet the matching conditions makes it impossible to construct a classical solution in the entire first quadrant. In the case when the initial data are insufficiently smooth, a mild solution of the initial-boundary value problem is constructed, and its uniqueness is proved.

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ОРИГИНАЛЬНАЯ СТАТЬЯ

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Начально-граничная задача с условиями Дирихле и Вентцеля для слабо квазилинейного биволнового уравнения

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Аннотация

Для нестрого гиперболического слабо квазилинейного биволнового уравнения, заданного в первом квадранте, рассматривается начально-граничная задача, в которой на пространственной полупрямой заданы условия Коши, а на временной полупрямой – условия Дирихле и Вентцеля. Решение строится в неявном аналитическом виде как решение некоторых интегро-дифференциальных уравнений. Методом продолжения по параметру исследуется разрешимость этих уравнений. Для рассматриваемой задачи доказывается единственность решения и установлены условия существования ее классического решения. Если данные задачи недостаточно гладкие, то строится слабое решение.

Ключевые слова: метод характеристик, слабо квазилинейное биволновое уравнение, нелинейное уравнение, нестрого гиперболическое уравнение, начально-краевая задача, условия согласования, классическое решение, метод продолжения по параметру, слабое решение.

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